Mathematics underachievers learning spatial knowledge

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When we think of mathematics curricula we generally think of doing things with numbers, that is, arithmetic. The concepts of space and spatial knowledge generally take a second place. Indeed, when you look at how we teach spatial and geometric concepts you see that we use arithmetic knowledge to describe and explain spatial properties and relationships.

Spatial knowledge and mathematics underachievement

The relegation of spatial knowledge to a subordinate position has particular implications for examining mathematics underachievement. First, most of use have observed in our teaching that some mathematics underachievers (that is, those who have difficulty comprehending numerical ideas) can comprehend elementary spatial concepts but because they lack fluency in number they cannot make sense of the relationships to be developed. Second, you have probably observed that other more able mathematics students (that is, those who do not have difficulty comprehending numerical ideas) have difficulty comprehending spatial concepts and are secure learning spatial knowledge only when the ideas are expressed in numerical terms.

This relegation does not make spatial knowledge less important or relevant. Indeed, we use our knowledge of spatial concepts to construct knowledge in other areas of mathematics. In this article we examine the acquisition of spatial knowledge by mathematics underachievers.

What do we mean by spatial knowledge ?

What do we mean by spatial knowledge ? When you hear the term 'spatial knowledge' you probably think of concepts such as up, straight, circle, angle, solid. Generally, spatial knowledge in mathematics refers to shapes and their properties. We think of spatial knowledge as consisting of two fundamental aspects

(1) shapes, for example, the number of sides they have, the number and size of angles, and

(2) position or location in space, for example, above, horizontal, etc.

In real-life use, on the other hand, 'space' is used to refer to capacity (for example, "How much space is in the room ?") or to a region that is empty (for example, "Leave a space" or "Sit in that space"). It is important, of course, when we are talking about space with underachievers, that we ensure that they understand what we mean by 'space'.

A related term is 'geometry'. This word has as its origin 'measurement of the earth'. The concepts that we currently examine in geometry have little to do with the earth. Generally it is used to refer to spatial concepts and relationships. Some definitions restrict it to two-dimensional space while others use a broader definition.

Using spatial knowledge for learning other mathematics knowledge

There are two types of spatial knowledge in mathematics curricula; knowledge of spatial relationships in the real world and knowledge of how spatial relationships are used to represent real world ideas

We use spatial knowledge to represent mathematics concepts in other areas. In our Western culture we impose a one dimensional or straight line model on a variety of phenonema, such as on time (time lines, digital time calendars, etc), on sets of numbers (as in the number line, graphs, etc), on natural phenonema such as height, distance, temperature and in representing statistical data. Clockfaces and price lists use spatial properties to represent ideas.

We also represent spatial properties using spatial means. An architect will build a three-dimensional model to display the spatial relationships in an intended construction. We would find it difficult to move around our cities and states without access to maps. The maps, of course, being two-dimensional, distort the three-dimensional world in which we live. We are generally, however, able to deal with these distortions; we do not become 'fussed' or confused by a hill or a slight curve that has not been shown on a map that we are following. Sometimes we draw pictures of real-life situations to make the spatial properties of the situation more obvious or clear; for example, to show that the roof of a house is rectangular. Many students, of course, have difficulty 'seeing' spatial relationships in real-life situations; they cannot generalize or map their spatial knowledge over the real-life situation. They have difficulty seeing a similarity between the clear black-line drawings that they handled in mathematics lessons and real-life situations.

We use relative spatial position to comprehend quantity and numbers in a variety of ways. A more numerous quantity is often assumed to be the one that covers the greatest space. At an early age children can mis-interpret this assumption to associate the overall spread or distribution of a quantity with its numerosity. Our notational system for numbers, the place value convention is based on a linear model. As well, we use spatial reference to describe whole number algorithms ("Write this number above that one").

The types of spatial difficulties displayed by underachievers

The types of spatial difficulties displayed by underachievers include the following;

(1) difficulties dealing with **directional concepts** such as up and down, right and left, etc. Many learning disabled children have diffculty comprehending these basic directional concepts. This has been associated with delayed perceptual-motor development (Kephart, 1971) and with laterality and directionality resolution difficulties.

(2) difficulties with the **elementary spatial concepts** examined by tests such as the Boehm Basic Concepts Test. Learning disabled students characteristically experience dealy in the acquisition of cocepts such as 'inside', 'above' and 'under', etc.

(3) **a spatial concept is associated with inappropriate criteria**, for example, the child believes that a change in position or size means a change in shape or concept;

-	*This is a right angled triangle*	*This isn't a right -angled triangle*	*These lines are parallel*	*These lines are not parallel*

Some children can recognize shapes only in 'standard' positions. They learn these as prototypes and have difficulty modifying these and seeing how transformations are still examples of the concept.

(4) the child cannot '**act mentally' on a shape or visualize it being changed** or transformed, for example, in the following the child cannot rotate a shape to produce another

Is a square *Isn't a square*

(5) the **concept of angle and the extent of rotation** cause difficulty: the child uses inappropriate information to make decisions about angles, for example, decide that one angle is larger than another because its arms are longer;

this angle is larger than this one

(6) the child uses i**nappropriate perceptual features** to categorize shapes

(7) the child has difficulty **representing 3-dimensional objects in 2-dimensions** and difficulty seeing 3-dimensional properties in 2-dimensional models and diagrams. The child has difficulties seeing a three-dimensional situation from different perspectives.

Why do children form misconceptions about spatial concepts ?

Misconceptions about spatial concepts can be attributed to a number of sources, for example

(1) perceptual difficulties are frequently associated with learning disabilities; these students may have difficulties integrating the components or parts of a spatial stimulus to form the whole, difficulties discriminating between the main visual information and irrelevant background information,

(2) lack of earlier sensory-motor experiences, such as building, matching, shape manipulatiion, etc.,

(3) difficulties learning visually or tactually; some children prefer to learn auditorily. When this is an extreme learning style, the children may have difficulty learning visually. This is the stimulus modality through which spatial information is generally provided, and

(4) inadequate teaching. According to Dickson, Brown and Gibson (1984), many of the misconceptions that children form about spatial concepts are due primarily to inadequate teaching.

One might expect therefore, that pupils have may have difficulty acquiring spatial knowledge because they have difficulty perceiving spatial patterns, processing spatial data and forming spatial concepts, as well as inadequate teaching.

It is important, therefore, to look at how children go about acquiring spatial knowledge as a basis for developing effective teaching.

How is spatial knowledge acquired ?

The acquisition of spatial concepts is an essential aspect of human development. Major theories of the acquistion of spatial knowledge have been provided by Piaget and by van Hiele. In this section we examine each of these in terms of light that they can throw on underachievement in thos area.

Piaget's theory of spatial development. Piaget (Piaget and Inhelder, 1967) completed his first major study of the acquisition of spatial knowledge in 1948. The approach taken is an aspect of his more general study of cognitive development. Piaget proposed that spatial knowledge is developed as children interact with their environment. The development of their spatial knowledge follows the sequence topological to projective to Euclidean. In other words, children progressively differentiate between geometric properties. The developmental progression is as follows:

(1) the child first notes global properties that are independent of size and shape; these are the topological properties. The child's drawing of a person may show features such as separation (the arms and body don't overlap), order (the two eyes are above the nose), enclosure (eyes, mouth, etc., are enclosed within the head) and continuity (the arms and legs are continuous with the body).

(2) the child becomes able to predict how an object will appear when it is seen from different perspectives; these are the projective properties. The child may for example, be able to imagine what a large tank will look like from above.

(3) the child learns the geometrical properties that relate to size, distance and shape, thus leading to differences between shapes based on spatial properties such as the size of angles, the number of parallel sides; these are the Euclidean properties.

Thus, children can comprehend a spatial concept such as triangle at a number of different levels

(1) recognize and label an instance of a triangle but not be able to explain why or to descibe its properties,

(2) recognize the triangular shape in different contextst

(3) recognize instances of the concept, descibe what they have in common and how they differ from four sided shapes, etc.

Suppose a child saw the following set of shapes; a square with a diamond inside it, a triangle, a square and a circle. The order in which the developing child would be expected to discriminate between these according to Piaget would be as follows;

(1) first the square with a diamond inside it would be discriminated from the others,

(2) second the circle is discriminated from the straight-sided shapes and

(3) third the straight-sided shapes are discriminated.

There are as well other key aspects of this development of spatial knowledge. First, the distinction is made between the perception and representation of spatial knowledge. Perception is the knowledge of objects that one obtains as a result of direct interaction with them. This begins when children first become interested in their world during the sensori-motor stage. Representation refers to the child's capacity to reason about the spatial properties of an object when it is no longer present and to think about spatial concepts without reference back to specific objects. This was seen as a form of mental imagery and was believed to begin to develop towards the end of the sensori-motor stage (around the age of two) and becomes refined towards the beginning of the concrete-operational stage.

A second aspect of Piaget's theory is the distinction made between two components of thought; the figural and the operative. The figurative aspect refers to fixed states; the child describes the spatial concept the way it is experienced. The operative aspects refer to mental operations that the child may apply to a spatial concept, for example, transforming between different states of a concept, predicting outcomes of an intended transformation. A child who can see a square and names it is using figurative knowledge. A child who can reason that the outcome of rotating a square would be a diamond is using operative knowledge.

Problems with Piaget's theory. Various difficulties have been noted with Piaget's model. The distinction between perception and visual imagery would seem to be less clear in child development than Piaget would suggest. As noted by Dickson, Brown and Gibson (1984), two-year old children show evidence of elementary mental visualization. These investigators review a range of empirical studies that suggest identify difficulties with Piaget's theory.

van Hiele's theory of spatial knowledge. The second theory of spatial development has been proposed by van Hiele . Van Hiele's theory proposes five levels of development:

(1) Level 1 - shapes are distinguished in their overall or global appearance and not on the basis of relationships between the the number or length of sides or angles, etc. A child operating at this level may be able to name or reproduce a square, a parallelogram and a triangle, but not see the relationships between them or their differences. They are simply separate and different shapes.

(2) Level 2 - an awareness of parts of shapes begins to develop, as children play with shapes in various ways they note individual properties of particular shapes. The child may, for example, note that a triangle has three sides and three angles and that a rectangle has four sides, that opposite sides are equal in length and also has four 'straight' angles. It is unlikely at this level that children relate different shapes. They extract these ideas through practical exploration, in play and in drawing activities.

(3) Level 3 - children begin to organize Level 2 findings into relationships between shapes. They may form the generalizations that as the number of sides a shape has gets bigger, so does the number of angles, that every square is also a rectangle and that every rectangle is also a parallelogram, that four-sided shapes can be made from two triangles and that five sides can be made from three triangles, etc. They learn these relationships through practical experiences, for example, they put together two, three and four triangles together and note the number of sides the shape has each time;

Levels 2 and 3 constitute the descriptive levels; children learn to recognize and manipulate geometric and spatial concepts in terms of their properties.

(4) Level 4 - children develop statements or 'child-propositions' to deduce one spatial property from another, for example, if the sum of the angles in a triangle is 180 degrees then it follows that if the total number of degrees in a shape is 180 degrees, the shape must be a triangle. When a line cuts across two parallel lines, the alternate angles on either side of the cutting line are equal. Similarly, when a line cuts across

two other lines and produces alternate angles that are equal, the two lines must be parallel. Children at this level can relate ideas but they do not recognize or comprehend the need for the rigorous proofs of geometric relationships characterized by formal treatments.

(5) Level 5 - children learn the more abstract aspects of deductive reasoning to prove geometric relationships. This is the theoretical level of understanding spatial concepts. Students can manipulate spatial ideas in an abstract way without needing to refer back to concrete or pictorial referents. Children here can devise formal geometric proofs and understand the processes involved to do this.

Levels 4 and 5 constitute the theoretical levels. Van Hiele proposed a sequence of learning phases to assist pupils to improve how they reason about spatial ideas. Many underachievers in this area hav difficulty progressing beyond level 1 thinking. This sequence provides a means for helping pupils to develop. Each learning peried builds on and extends the preceding level. The instruction makes explicit what was only implied at the preceding level of thought. Language has an important role to play in learning. Each level has a vocabulary that is used to represent the concepts and relationships. New language is introduced at each learning period to discuss the new ideas. As well, students at a lower level are not expected to understand ideas demanding a higher level of thought. The following sequence moves pupils from the direct instruction to the student's understanding independent of the teacher;

(1) inquiry; the teacher engages pupils in two-way discussions about the spatial ideas to be learnt. The teacher learns how the pupils interpret the words and guides them to construct an understanding of the topic being studied.

(2) directed orientation; the teacher sequences activities for guided pupil exploration, leading them to become familiar with the characteristic structures.

(3) explication; the students build on their foregoing experiences to refine their comprehension of the topic being examined and express their ideas and understandings.

(4) free orientation; the students develop their own procedures for solving longer, more complex spatial problems. This allows them to identify many of the relations between the spatial ideas being learnt.

(5) integration; the students review their findings and form an overview. The relationships are unified into a new domain of thought.

Support for van Hiele's model of spatial learning has been provided by many investigators (for reviews see Hoffer, 1983).

What behaviours are indicative of understanding at the various levels ? The following is an example. Many late-primary level students, while familiar with spatial terms such as 'triangle', have little understanding of what they mean, for example, they may rotate the drawing of a triangle to try to see it in the 'textbook position', or they amy say that a scalene triangle is "too tall / too flat... to be a triangle". Teachers can use the van Hiele model to analyse how they are teaching spatial ideas.

Do males learn spatial ideas more easily than females ?

The importance of spatial knowledge to the learning of mathematics more generally has been discussed above. Several investigators recently have proposed that males and females differ in how they think about spatial concepts, and that this difference accounts in part for the better mathematics performance by males in the early - middle secondary years.

The main spatial ability to receive a focus is spatial visualization, the ability to mentally manipulate, rotate, twist, or invert a pictorially presented stimulus (Fennema & Tartre, 1985) and to generate, retain and transform abstract images (Kyllonen, Lohman & Snow 1984).

Fennema (1075) reported that gender differences in spatial visualization emerge between upper promary and secondary level and parallel sex differences in mathematics achievement. This difference was observed to affect mathematics performance differentially, such that females seem to convert spatial abilities into mathematics achievement to a greater extent than males (Ethington & Wolfle, 1984). Fennema & Tartre (1985) in a longitudinal study of grade 6 to 8 students, monitored how students who differed in their verbal and spatial performance used spatial visualization skills to to solve word and fraction problems. Several findings were noted

(1) although students who differed in their spatial visualization and verbal skills also differed in how the went about problem-solving, they didn't differ in their ability to solve the problems,

(2) students who had a higher level of spatial visualization were more likely to use spatial skills in problem-solving

(3) boys solved more problems than girls and while the boys who solved the most were also low in spatial visualization and high in verbal skill, the girls who were low in spatial visualization and high in verbal skill solved least problems. This suggested that females may be more debilitated than males by low spatial visualization.

(4) while girls reported using pictures more than boys did, the boys had more pictorial information when they solved problems. As well, the girls verbalized more complete relevant information than they boys, but this verbalization didn't lead to more correct solutions.

In an overview of these findings, Fennema & Tartre note that all of the gender differences that they observed were small, particularly in the light of the much larger spread within each gender group. While the use if spatial visualizatoin may assist in explaining sex-related differences in mathematics, these investigators counsel against the conclusion that all girls are less able than all boys to use their spatial visualization skills than boys.

A sequence for teaching spatial knowledge.

Students who have difficulty learning spatial knowledge can be assisted by the following sequence:

(1) students manipulate shapes in free play situations, such as building, solving spatial problems, shape post-boxes, drawing two and three dimensional objects explore shapes through physical actions, etc.

(2) students recognize and name individual shapes such as

Students practise describing individual shapes, learn to use distinctive spatial properties to describe individual shapes, learn to classify shapes using spatial labels.

(3) students analyse the characteristic properties of individual simple regular shapes on the basis on the number of sides and angles, etc. They describe differences between shapes, draw shapes and visualize shapes.

(4) students manipulate groups of shapes, describe shapes from different perspectives, fit shapes together,

(5) students generate more general spatial concepts by physical actions, such as

(a) all squares are rectangles

(b) the set of polygons.

(6) students recognize spatial concepts in different perceptual contextx, for example, they act on one shape to produce the other and discuss the effect of particular transformations. Gradually they are encouraged to visualize these types of actions.

Modern techniques and procedures for helping underachievers to learn spatial knowledge

Over the past two decades a range of modern techniques and materials have been developed for assisting pupils to learn spatial knowledge. These include geoboards and Logo. Many of these materials are particularly useful for underachievers because they encourage physical action, visualization, etc.

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